On the properties of Circular-Beams: normalization, Laguerre-Gauss expansion and free-space divergence

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Circular-Beams were introduced as a very general solution of the paraxial wave equation carrying Orbital Angular Momentum. Here we study their properties, by looking at their normalization and their expansion in terms of Laguerre-Gauss modes. We also study their far-field divergence and, for particular cases of the beam parameters, their possible experimental generation.

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Orbital Angular Momentum (OAM) [1] of light has recently attracted a lot of attention as a new promising resource for fundamental and applied physics [2–10], such as biophysics [2], microscopy [3], astronomy [4] and metrology [7]. In this framework, Circular-Beams (CiBs) were introduced in [11] as a very general solution carrying OAM of the paraxial wave equation. The importance of the CiBs is underlined by the fact that they represent a generalization of many well known beams with OAM. Indeed, for particular values of the beam parameters, the CiBs reduce to standard [12] or elegant Laguerre-Gauss [13], Hypergeometric-Gauss [14, 15], Hypergeometric [16] modes and others [11]. Then, the study of the CiB properties allows to better understand, in a unified way, the features of such well known beams. For instance, since the propagation of CiBs through ABCD optical systems can be easily described in terms of their defining parameters q_0 and q_1 [11], the same propagation law can be also applied for their known particular cases. Moreover, while Laguerre-Gauss (LG) beams seem a natural choice for describing OAM states in relatively simple terms, in several cases they only represent an approximation of the generated beam. Indeed, a special CiB quite different from an LG mode, is obtained by applying a singular phase factor $\exp(im\phi)$ to a Gaussian beam (see eq. (18) and [14]). This technique represents a very common and simple way to generate OAM light [5, 8, 10]. Thus, the study of the Circular-Beams gives insight for the precise modeling, generation and exploitation of OAM in classical or quantum applications.

In this letter, we analyze the properties of the CiBs: in particular, we will explicitly derive their expansion in terms of Laguerre-Gauss modes and their normalization constant, allowing the evaluation of their free-space divergence. We will also investigate the constraints on the values of the beam parameters arising by the request of square integrability and regularity at any finite point. Then, we will study the experimental generation of a subclass of CiBs. Our results will provide an useful tool for the complete modeling of CiBs, paving the way towards their experimental implementation.

CiBs were introduced in [11], without the evaluation of their normalization. As a first result of this letter, we give the detailed expression of the normalized CiB. Let's define the complex parameters $q(z), \widetilde{q}(z)$, ξ and the scale factor $\chi(z)$ as:

$$
\begin{aligned}
q(z) &= z + q_0\,, & \xi &= \frac{q_1 - q_0}{q_0^* - q_1}\,,\\
\widetilde{q}(z) &= z + q_1\,, & \frac{1}{\chi^2(z)} &= \frac{ik}{2}\left[\frac{1}{q(z)} - \frac{1}{\widetilde{q}(z)}\right]\,,
\end{aligned}
\tag{1}
$$

were the real and imaginary part of the two complex beam parameter q_0 and q_1 are given by $q_0 = iz_0 - d_0$, $q_1 = iz_1 - d_1$ being $z_0 = \pi W_0^2/\lambda$ and d_0 the analogues of the confocal parameter and the location of the waist for a Gaussian beam. The normalized monochromatic CiB in cylindrical coordinates $\mathbf{x} \equiv (r, \phi, z)$ and propagating along the z axis is defined as $\exp[i(\omega t - kz)]\mathrm{CiB}_{p,m}^{(q_0,q_1)}(\mathbf{x})$ where

$$
\begin{aligned}
\mathrm{CiB}_{p,m}^{(q_0,q_1)}(\mathbf{x}) = &\left(i\sqrt{2}\frac{z_0}{W_0}\right)^{|m|+1} \left[\pi\, |m|!\, \Psi_{p,m}^{(\xi)}\right]^{-\frac{1}{2}} \times\\
&\frac{e^{-\frac{ikr^2}{2q(z)}}}{q(z)} \left[(1+\xi)\frac{\widetilde{q}(z)}{q(z)}\right]^{\frac{p}{2}} \left[\frac{r}{q(z)}\right]^{|m|} \times\\
&{}_1F_1\left(-\frac{p}{2}, |m|+1; \frac{r^2}{\chi^2(z)}\right)e^{im\phi}\,.
\end{aligned}
\tag{2}
$$

In the above equation ${}_1F_1$ is the confluent Hypergeometric function ${}_1F_1(a, b; x) = \sum_{n=0}^{+\infty} \frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)} \frac{x^n}{n!}$ and $\Psi_{p,m}^{(\xi)}$ a normalization factor given by

$$
\begin{aligned}
\Psi_{p,m}^{(\xi)} &= \sum_{n=0}^{+\infty} \frac{\Gamma(n - \frac{p}{2})\Gamma(n - \frac{p^*}{2})|m|!}{\Gamma(-\frac{p}{2})\Gamma(-\frac{p^*}{2})n!(|m|+n)!}|\xi|^{2n}\\
&\equiv {}_2F_1\left[-\frac{p}{2}, -\frac{p^*}{2}, 1+|m|, |\xi|^2\right]\,.
\end{aligned}
\tag{3}
$$

The parameters $m \in \mathbb{Z}$ and $p \in \mathbb{C}$ respectively represent the amount of OAM carried by the beam and the analogous of the Laguerre-Gauss mode radial index. As noted in [11], the field is invariant (up to a normalization factor) by the transformation $(p, m, q_0, q_1) \to (-p - 2|m| - 2, m, q_1, q_0)$[1]. As we will see, to guarantee the square integrability and regularity at each finite z

[1] In [11] the field was defined with p replaced by $i\gamma - |m| - 1$

and r, some constraints will arise on the allowed values of p.

As a second result of this letter, we give the expansion of the CiB in terms of the Laguerre-Gauss beams $\mathrm{LG}_{n,m}(\mathbf{x})$ when $q_1 \neq q_0^*$ and $\Im\mathrm{m}(q_0) > 0$:

$$\mathrm{CiB}_{p,m}^{(q_0,q_1)}(\mathbf{x}) = \sum_{n=0}^{+\infty} A_{p,n}^{(m,\xi)} \mathrm{LG}_{n,m}(r,\phi,z-d_0)\,, \qquad (4)$$

where

$$A_{p,n}^{(m,\xi)} = \frac{\xi^n}{\sqrt{\Psi_{p,m}^{(\xi)}}} \frac{\Gamma(n-\frac{p}{2})}{\Gamma(-\frac{p}{2})} \sqrt{\frac{|m|!}{n!(|m|+n)!}}\,. \qquad (5)$$

The above expansion does not hold when $q_1 = q_0^*$ and $\Im\mathrm{m}(q_0) > 0$: indeed, in this case, the $\mathrm{LG}_{n,m}$ mode is directly obtained as a particular CiB, namely $\mathrm{LG}_{n,m}(r,\phi,z-d_0) = (-1)^n \mathrm{CiB}_{2n,m}^{(q_0,q_0^*)}(\mathbf{x})$. Equation (4) implicitly shows that the CiBs satisfy the paraxial wave equation since they can be represented as a linear combination of its solutions (i.e. the LG modes). For completeness, we here report the explicit expression of the standard Laguerre-Gauss modes:

$$\mathrm{LG}_{n,m}(\mathbf{x}) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{n!}{(|m|+n)!}} \frac{e^{-\frac{ikr^2}{2q(z)}}}{W(z)} \left[\frac{\sqrt{2}r}{W(z)}\right]^{|m|} \times L_n^{(|m|)}\left[\frac{2r^2}{W^2(z)}\right] e^{im\phi} e^{i(2n+|m|+1)\zeta(z)}\,, \qquad (6)$$

where n, $m \in \mathbb{Z}$ with $n \geq 0$, $L_n^{(|m|)}(x)$ is the generalized Laguerre polynomial, $W(z) = W_0\sqrt{1+(z/z_0)^2}$ the beam size and $\exp[i\zeta(z)] = (z_0+iz)/|z_0+iz|$ the Gouy phase.

We now demonstrate eqs. (3) and (4). When $q_1 \neq q_0^*$ and $\Im\mathrm{m}(q_0) > 0$ we may exploit the expansion of the Hypergeometric function in terms of Laguerre polynomials:

$${}_1F_1(a,b,\frac{YZ}{Y-1}) = \sum_{n=0}^{+\infty} \frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)} \frac{Y^n L_n^{(b-1)}(Z)}{(1-Y)^{-a}}. \qquad (7)$$

Indeed, by using $Z = \frac{2r^2}{W(z-d_0)^2}$ and $Y = -\xi\frac{q(z)^*}{q(z)}$ in the above equation, relation (4) can be easily obtained. Regarding the value of the constant $\Psi_{p,m}^{(\xi)}$ (see eq. (3)), it can be determined directly from the expansion given in eq. (4). By recalling that the Laguerre-Gauss modes are orthonormal with respect to the scalar product $\int \mathrm{d}\phi \int \mathrm{d}r\, r\, \mathrm{LG}_{n_1,m_1}^*(\mathbf{x})\mathrm{LG}_{n_2,m_2}(\mathbf{x}) = \delta_{n_1,n_2}\delta_{m_1,m_2}$, the CiB is easily found to be normalized to 1 if the constant $\Psi_{p,m}^{(\xi)}$ is defined as (3).

As already anticipated, to obtain square integrable and regular beams, some constraints (summarized in eq. (8)) arises on the allowed values of p. Let's start by the square integrability, equivalent to the convergence of the sum in eq. (3). The latter converges to the Hypergeometric function ${}_2F_1$ in three cases: a) when $\Im\mathrm{m}(q_0) > 0$, $\Im\mathrm{m}(q_1) > 0$, $\forall p$; b) when $\Im\mathrm{m}(q_0) > 0$, $\Im\mathrm{m}(q_1) = 0, +\infty$ and $\Re\mathrm{e}(p) > -1-|m|$; c) when $\Im\mathrm{m}(q_0) > 0$, $\Im\mathrm{m}(q_1) < 0$ and $p = 2\ell$ with $\ell \in \mathbb{N}$. In the latter case the sum becomes finite, since in eq. (4) the expression $\sum_{n=0}^{+\infty}\Gamma(n-\frac{p}{2})/\Gamma(-\frac{p}{2})$ must be replaced by $\sum_{n=0}^{\ell}(-1)^n\ell!/(\ell-n)!$. The three cases above cited respectively correspond to $|\xi| < 1$, $|\xi| = 1$ and $|\xi| > 1$. As noticed in [11], the same constraints can be obtained by looking at the CiB behavior at large r.

However, further conditions on p arise by requiring the regularity of the field at each finite z. In the cases a) and c) the field is regular at any finite r and z. On the other hand, when $\Im\mathrm{m}(q_0) > 0$ and $\Im\mathrm{m}(q_1) = 0$ the field may be singular at $z = d_1$. Indeed, since in this case we have $\lim_{z\to d_1}\mathrm{CiB}_{p,m}^{(q_0,q_1)}(\mathbf{x}) \propto r^{p+|m|}$, regularity in $r = 0$ requires $\Re\mathrm{e}(p) \geq -|m|$. Then, the integrability and the absence of singularity at any finite r and z are only guaranteed in the following four inequivalent cases:

$$\begin{aligned} &\mathrm{I})\ \Im\mathrm{m}(q_0) > 0\,,\ \Im\mathrm{m}(q_1) = +\infty\,,\ \Re\mathrm{e}(p) > -|m|-1 \\ &\mathrm{II})\ \Im\mathrm{m}(q_0) > 0\,,\ \Im\mathrm{m}(q_1) < 0\,,\ p = 2\ell \text{ with } \ell \in \mathbb{N} \\ &\mathrm{III})\ \Im\mathrm{m}(q_0) > 0\,,\ \Im\mathrm{m}(q_1) = 0\,,\ \Re\mathrm{e}(p) \geq -|m| \\ &\mathrm{IV})\ \Im\mathrm{m}(q_0) > 0\,,\ \Im\mathrm{m}(q_1) > 0\,,\ \forall p \in \mathbb{C} \end{aligned} \qquad (8)$$

Other equivalent cases may be obtained by the symmetry $(p,m,q_0,q_1) \to (-p-2|m|-2,m,q_1,q_0)$.

To further investigate the properties of the CiBs, we now evaluate their free-space divergence, an important parameter for experimental implementation and propagation through long distances. For general light beam, the root mean square (rms) divergence θ_{rms} can be defined as [17–19]

$$\theta_{\mathrm{rms}} \equiv \lim_{z\to+\infty}\frac{\sigma_{\mathrm{rms}}(z)}{z}\,, \qquad (9)$$

where $\sigma_{\mathrm{rms}}^2(z)$ is the second moment of the intensity $I(\mathbf{x})$:

$$\sigma_{\mathrm{rms}}^2(z) = \int_0^{2\pi}\mathrm{d}\phi\int_0^{+\infty}\mathrm{d}r\, r\left[r^2\, I(\mathbf{x})\right]\,. \qquad (10)$$

By exploting the expansion of the CiB in terms of the LG modes (4) and the relation $2\int\mathrm{d}\phi\int\mathrm{d}r\, r^3\,\mathrm{LG}_{\ell,m}(\mathbf{x})\mathrm{LG}_{n,m}^*(\mathbf{x}) = W^2(z)[B_{m,\ell}\delta_{n,\ell} - C_{m,\ell}\delta_{n+1,\ell} - C_{m,n}^*\delta_{n,\ell+1}]$ with $B_{n,m} = |m|+2n+1$ and $C_{m,n}(z) = \sqrt{n(|m|+n)}\exp[2i\zeta(z)]$ it is possible to give an explicit expression for $\sigma_{\mathrm{rms}}^2(z)$, namely:

$$\frac{2\,\sigma_{\mathrm{rms}}^2(z+d_0)}{W^2(z)} = 1+|m|+\Phi_{p,m}^{(\xi)}+\Re\mathrm{e}[X(z)]\,, \qquad (11)$$

where $X(z) = \xi(p-\Phi_{p,m}^{(\xi)})\exp[2i\zeta(z)]$ and

$$\Phi_{p,m}^{(\xi)} = \frac{|p\,\xi|^2}{2+2|m|}\frac{{}_2F_1[1-\frac{p}{2},1-\frac{p^*}{2},2+|m|,|\xi|^2]}{{}_2F_1[-\frac{p}{2},-\frac{p^*}{2},1+|m|,|\xi|^2]}\,. \qquad (12)$$

By plugging the above result into (9), the rms divergence of the CiBs may be expressed as

$$\theta_{\mathrm{rms}} = \theta_0\sqrt{1+|m|+\Phi_{p,m}^{(\xi)}-\Re\mathrm{e}\left[\xi(p-\Phi_{p,m}^{(\xi)})\right]}\,. \qquad (13)$$

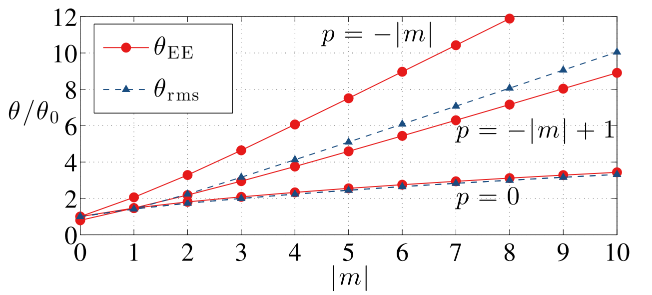

FIG. 1. Divergences θ_{rms} and θ_{EE} in function of $|m|$ for the CiB with $q_1=\Re\mathrm{e}(q_0)$ and $p=0$, $p=-|m|+1$ or $p=-|m|$. When $p=-|m|$, only θ_{EE} can be defined. The beam is only defined for integer values of m and lines represent just a guide for the overall trends. Note that when $p=0$ the $\mathrm{CiB}_{p,m}$ reduces to the Laguerre-Gauss mode $\mathrm{LG}_{0,m}$.

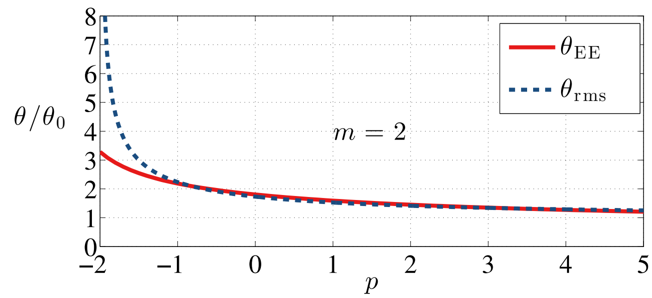

FIG. 2. Divergences θ_{rms} and θ_{EE} in function of p for the CiB with $\Im\mathrm{m}(p)=0$, $q_1=\Re\mathrm{e}(q_0)$ and $m=2$. When p is approaching to $-|m|$, the value θ_{rms} tends to infinity.

In the above equation $\theta_0=W_0/(\sqrt{2}z_0)$ is the divergence of a Gaussian Beam (i.e. a CiB with $p=m=0$) with confocal parameter z_0 and beam waist $W_0=\sqrt{\lambda z_0/\pi}$. As we will show in the following (see eq. (17)), the divergence θ_{rms} is not finite when $\Im\mathrm{m}(q_1)=0$ and $\Re\mathrm{e}(p)=-|m|$. Indeed, for such beams, at fixed z and large r, the intensity fall-off as r^{-4}. While the field is thus square integrable, the integral (10) is divergent and the $\sigma^2_{\mathrm{rms}}(z)$ cannot be defined.

To avoid such problem, we may use an alternative definition of the divergence in terms of the so-called encircled-energy. It requires the determination of the radius $R_{\mathrm{EE}}(z)$ whose corresponding circle centered on the beam axis contains a given amount of beam energy. Then, $R_{\mathrm{EE}}(z)$ must be calculated by the implicit relation $\int \mathrm{d}\phi\int_0^{R_{\mathrm{EE}}(z)}\mathrm{d}r\, r I(\mathbf{x})=I_0$, with I_0 a fixed constant. Given $R_{\mathrm{EE}}(z)$, the corresponding divergence θ_{EE} can be determined similarly to eq. (13):

$$\theta_{\mathrm{EE}}=\lim_{z\to+\infty}\frac{R_{\mathrm{EE}}(z)}{z}\,. \tag{14}$$

If we choose $I_0=1-1/e$ (chosen to achieve $\theta_{\mathrm{EE}}=\theta_{\mathrm{rms}}=\theta_0$ for the Gaussian beam), the divergence θ_{EE} of a CiB can be obtained by the following implicit relation:

$$\int_0^{(\theta_{\mathrm{EE}}/\theta_0)^2} e^{-t}t^{|m|}\left|H_{p,m}(\frac{\xi t}{\xi+1})\right|^2 dt=I^{(\xi)}_{p,m}\,, \tag{15}$$

with

$$I^{(\xi)}_{p,m}=\frac{e-1}{e}\frac{|m|!\,\Psi^{(\xi)}_{p,m}}{|(1+\xi)^p|}\,, \tag{16}$$

and $H_{p,m}(z)$ a shorthand for the Hypergeometric function ${}_1F_1(-p/2,|m|+1,z)$. Relation (15) is obtained by inserting the expression of the CiB (2) into the integral defining $R_{\mathrm{EE}}(z)$, by changing variable into $r=W(z)\sqrt{t/2}$, by taking the limit for $z\to+\infty$, and by using the property $\lim_{z\to+\infty}\sqrt{2}R_{\mathrm{EE}}(z)/W(z)=\theta_{\mathrm{EE}}/\theta_0$.

To better understand the properties the CiBs, we now analyze their features for some particular values of the beam parameters and study in more detail their divergences and their experimental generation. Indeed, we would like to examine the CiBs with $|\xi|=1$ corresponding to $\Im\mathrm{m}(q_1)=0$ or $\Im\mathrm{m}(q_1)\to\infty$. In this case, a simpler expression for the CiB is obtained. Indeed, for $|\xi|=1$ we have $2\,\Phi^{(\xi)}_{p,m}=|p|^2/[\Re\mathrm{e}(p)+|m|]$ and the rms divergence and normalization factor simplify to

$$\theta_{\mathrm{rms}}=\theta_0\sqrt{1+|m|-\Re\mathrm{e}(p\,\xi)+\frac{|p|^2}{2}\frac{\Re\mathrm{e}(\xi)+1}{\Re\mathrm{e}(p)+|m|}}\,,$$
$$\Psi^{(\xi)}_{p,m}=|m|!\,\frac{\Gamma(|m|+1+\Re\mathrm{e}(p))}{|\Gamma(|m|+1+\frac{p}{2})|^2}\,. \tag{17}$$

As it is evident from (17), when $\Im\mathrm{m}(q_1)=0$ and $\Re\mathrm{e}(p)=-|m|$ the rms divergence cannot be defined, as already anticipated. For such beams, the divergence can be only evaluated by θ_{EE}. To better illustrate the behavior of the divergences, we show in Figure 1 the values of θ_{rms} and θ_{EE} in function of m for the CiB with $q_1=\Re\mathrm{e}(q_0)$, such that $\xi=1$ and $\theta_{\mathrm{rms}}=\theta_0\sqrt{1+m^2/(p+|m|)}$. We show three different CiBs classes, corresponding to the p parameter given by $p=0$, $p=-|m|+1$ or $p=-|m|$. In the latter case ($p=-|m|$), as already discussed, only θ_{EE} can be shown since θ_{rms} in not defined. The case $p=0$ corresponds to the LG mode $\mathrm{LG}_{0,m}$. Similarly, in Figure 2 we show the two divergences θ_{rms} and θ_{EE} in function of p for the CiB with $\Im\mathrm{m}(p)=0$, $q_1=\Re\mathrm{e}(q_0)$ and $m=2$. It is worth noticing that increasing the value of p will reduce both divergences such that $\lim_{p\to+\infty}\theta=\theta_0$.

The role of the imaginary part of the p parameter can be appreciated when $q_1=-d_1$. In such case, at fixed $\Re\mathrm{e}(p)$, the rms divergence is minimized for $\Im\mathrm{m}(p)=(\Re\mathrm{e}(p)+|m|)\frac{d_1-d_0}{z_0}$ and becomes a decreasing function of $\Re\mathrm{e}(p)$, namely $\theta_{\mathrm{rms}}=\theta_0\sqrt{1+\frac{m^2}{\Re\mathrm{e}(p)+|m|}\frac{z_0^2}{z_0^2+(d_0-d_1)^2}}$. Then, when $-q_1=d_1\neq d_0$, a complex value of p is required to minimize the beam divergence.

We finally analyze a possible experimental generation of such subclass of CiBs. The cases $q_1=\Re\mathrm{e}(q_0)=-d_0$ and $\Im\mathrm{m}(q_1)\to\infty$ respectively correspond to the HyGG

and HyGG-II modes introduced in [14, 15]. The general case $q_1 = -d_1$, that we call generalized Hypergeometric-Gaussian beam (gHyGG), is particularly interesting from the point of view of experimental generation. Indeed, the behavior of the field for $z \to d_1$ gives a clear hint for its possible experimental generation. For $q_1 = -d_1$ we have

$$\lim_{z\to d_1} \mathrm{CiB}^{(q_0,q_1)}_{p,m}(\mathbf{x}) \propto e^{-\frac{ikr^2}{2(d_1-d_0+iz_0)}} \frac{r^{p+|m|}e^{im\phi}}{d_1-d_0+iz_0}\,. \quad (18)$$

This feature is very useful because the r.h.s. of eq. (18) can be generated by applying the singular phase factor $\exp(im\phi)$ and a polynomial transmittance profile of the order $p+|m|$ to Gaussian (TEM_{00}) beam at a distance $d_1 - d_0$ from its waist. In particular, the $p = -|m|$ modes are simply generated by applying the phase factor $\exp(im\phi)$ to a Gaussian beam at a distance $d_1 - d_0$ from its waist plane. Such phase factor can be experimentally implemented by q-plates [20] or phase-plates [21].

In the present letter we studied the properties of the Circular-Beams, a general solution of the paraxial wave equation with OAM depending on three complex parameters q_0, q_1 and p and an integer parameter m, related to the content of OAM. The allowed values of the beam parameters are given in eq. (8) and the corresponding ones obtained by the symmetry $(p, m, q_0, q_1) \to (-p-2|m|-2, m, q_1, q_0)$. We derived the normalization of the CiBs and their expansion in terms of Laguerre-Gauss modes, a standard OAM basis (see eq. (4)). Such expansion allowed us to study the divergence of the CiBs in free-space propagation. We defined the divergence by two methods, the rms (see eq. (13)) and the encircled-energy (see eq. (15)) and studied their behavior for particular values of the beam parameters. We finally suggested the experimental generation of a subclass of CiBs by looking at their behavior in a particular transverse plane, see (18). Our achievements, providing the estimation of the beam divergence for the CiBs (the most general beam with OAM known so far), may have potential application for OAM transportation in free space, such as multi-channel/multiplexing classical or quantum communication [6, 10, 22].

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